

Low Rank Separable States Are A Set Of Measure Zero Within The Set of Low Rank States

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Abstract

It is well known that the set of separable pure states is measure 0 in the set of pure states. We extend this fact and show that the set of rank r separable states is measure 0 in the set of rank r states provided r is less than $\prod_{i=1}^p n_i + p - \sum n_i$.

Recently quite a few authors have looked at low rank separable and entangled states. (See [1] and the references therein and [2] and [3] Therefore it makes sense to determine the size of the set of rank r separable states within the set of rank r states. For rank 1, it is well known that the separable states are a set of measure zero. This contrasts with the maximal rank case, where the separable states not only are not measure zero, but contain an open set.

The purpose of this note is to show that the rank 1 result is also true for many other low ranks. In particular, suppose we have p particles modelled on the Hilbert space $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_p}$. Then the following is true.

Theorem 1 *Let S_r be the set of rank r separable matrices on $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_p}$ and D_r the set of all rank r density matrices. S_r is measure 0 in D_r , for all $r < N + p - \sum n_i$, where $N = n_1 \dots n_p$*

This is an extension of recent results of Chen. In [3], he showed in the bipartite case where $H = \mathbb{C}^m \otimes \mathbb{C}^m$ that S_r is measure 0 in D_r if $r < 2m - 3$. As a result of Theorem 1, we see that S_r is in fact measure 0 in D_r if $r < m^2 - 2m + 2 = (m - 1)^2$. Since the number of possible ranks is m^2 , we see that in this case the proportion of those r for which S_r is measure 0 in D_r is $(1 - (1/m^2))^2$. In the case of p -qubits, S_r is measure 0 in D_r if $r < 2^p - 2p + p = 2^p - p$. Since the number of possible ranks in this case is 2^p , we see that the proportion of those r for which S_r is measure 0 in D_r is $1 - (p/2^p)$. Thus we see that the ranks of "low" rank matrices can be quite high.

The proof will use Sard's Theorem [4] to show the set of ranges of separable rank r density matrices is measure 0 in the set of ranges of rank r density matrices, i.e. within the set of r -dimensional subspaces of \mathbb{C}^N ; i.e. within

$G(N, r)$, the Grassmann manifold of r -planes in \mathbb{C}^N [5], if $r < N + p - \sum n_i$. That this is what we need to consider follows from the fact D_r is $G(N, r) \times \text{Herm}_1^+(r)$, where $\text{Herm}_1^+(r)$ is the space of Hermitian $r \times r$ matrices that are positive definite and trace 1. For those not familiar with Sard's Theorem, we will need only a simple consequence of it. Namely, we need the fact that if $f : M \rightarrow N$ is a smooth function (i.e. infinitely differentiable function) between an m -dimensional space M and an n -dimensional space N and if $m < n$, then $f(M)$ is measure zero in N . This should be intuitively obvious: the smooth image of a lower dimensional space in a larger one is measure zero. Think for instance of a smooth curve in a plane. The curve has 0 area.

Lemma 2 *Suppose A and B are positive semi-definite linear operators. Then $\text{Ker}(A + B) = \text{Ker}A \cap \text{Ker}B$ and $\text{Range}(A + B) = \text{Range}A + \text{Range}B$.*

Proof. Clearly $\text{Ker}A \cap \text{Ker}B \subset \text{Ker}(A+B)$ and $\text{Range}(A+B) \subset \text{Range}A + \text{Range}B$. Suppose $v \in \text{Ker}A \cap \text{Ker}B$. Then $0 = \langle (A+B)v, v \rangle = \langle Av, v \rangle + \langle Bv, v \rangle$. Since A and B are positive semi-definite, this means $Av = 0$ and $Bv = 0$, hence $\text{Ker}A \cap \text{Ker}B = \text{Ker}(A+B)$. Since $\text{Ker}(A+B)$ is the orthogonal complement of $\text{Range}(A+B)$, it follows that $\text{Ker}A \cap \text{Ker}B$ is the orthogonal complement of $\text{Range}(A+B)$. But $\text{Ker}A \cap \text{Ker}B$ is the orthogonal complement of $\text{Range}(A+B)$, so $\text{Range}(A+B) = \text{Range}A + \text{Range}B$. ■

If A is a separable density matrix, then A is the convex combination of projections onto product states. It follows from the lemma that the range of A therefore has a basis of product states [6]. The product states are precisely the image of $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}$ in \mathbb{P}^{N-1} under the map induced by tensor product on $\mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_p}$, where \mathbb{P}^k is k -dimensional complex projective space (i.e. the space of rays in \mathbb{C}^{k+1}).

As for the manifold of r -dimensional subspaces of \mathbb{C}^N , i.e., the Grassmann manifold, $G(N, r)$, it is obtained by first considering \mathbb{C}^{Nr} as being the set of $N \times r$ complex matrices and taking $L(N, r)$ to be the open subset consisting of those matrices with rank r . $G(N, r)$ is then the orbit space $GL(r, \mathbb{C}) \backslash L(N, r)$.

Let $\tilde{\mu} : \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1} \rightarrow \mathbb{P}^{N-1}$ be the map induced by tensor product and take $\mu : (\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})^r \rightarrow (\mathbb{P}^{N-1})^r$ to be $(\tilde{\mu}, \dots, \tilde{\mu})$. Let $Q = \mu^{-1}(\tilde{L}(N, r))$, where $\tilde{L}(N, r)$ is the image of $L(N, r)$ in $\mathbb{P}^{r(N-1)}$. Let $\tilde{\pi} : \tilde{L}(N, r) \rightarrow G(N, r)$ be the projection induced by $\pi : L(N, r) \rightarrow G(N, r)$. Then $\tilde{\pi} \circ \mu : Q \rightarrow G(N, r)$ has as its range the r -dimensional subspaces that have a basis of product states.

Since $\tilde{\pi} \circ \mu : Q \rightarrow G(N, r)$ is a smooth map, the theorem follows from Sard's theorem and the facts that $\dim G(N, r) = r(N - r)$ and $\dim(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})^r = r(\sum(n_i - 1)) = r(p - \sum n_i)$. Thus the $\dim Q < \dim G(N, r)$ if $r(p - \sum n_i) < r(N - r)$ that is if $r < N + P - \sum n_i$.

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References

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